

ON THE HIGHER-DIMENSIONAL HARMONIC ANALOG OF THE LEVINSON LOG LOG THEOREM.

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ABSTRACT. Let $M: (0, 1) \rightarrow [e, +\infty)$ be a decreasing function such that $\int_0^1 \log \log M(y) dy < +\infty$. Consider the set H_M of all functions u harmonic in $P := \{(x, y) \in \mathbb{R}^n : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < 1, |y| < 1\}$ and satisfying $|u(x, y)| \leq M(|y|)$. We prove that H_M is a normal family in P .

1. PRELIMINARIES.

Let P be a rectangle $(-a, a) \times (-b, b)$ in \mathbb{R}^2 and let $M: (0, b) \rightarrow [e, +\infty)$ be a decreasing function. Consider the set \mathcal{F}_M of all functions f holomorphic in P such that $|f(x, y)| \leq M(|y|)$, $(x, y) \in P$. The classical Levinson theorem asserts that \mathcal{F}_M is a normal family in P if $\int_0^b \log \log M(y) dy < +\infty$. We refer the reader to [4, 5, 6, 7, 8, 11, 15, 18, 19, 20, 21, 25, 26, 27, 30] for various proofs, history of the question and related topics. This statement is sharp, i.e. for regular (continuous and decreasing) majorants M the family \mathcal{F}_M is normal if and only if $\int_0^b \log \log M(y) dy < +\infty$ (see [18], p.379–383 and [4]).

A function $\log^+ x$ is defined by

$$\log^+ x = \begin{cases} \log x, & x \geq 1 \\ 0, & x \leq 1. \end{cases}$$

Our result is the following theorem, which extends the Levinson log log theorem for holomorphic functions to harmonic functions in \mathbb{R}^n , $n \geq 2$.

Theorem 1.1. *Let Ω denote the set $\{(x, y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < R, |y| < H\}$, where R and H are some positive numbers. Suppose a function $M: (0, H) \rightarrow \mathbb{R}_+$ is decreasing and*

$$(1) \quad \int_0^H \log^+ \log^+ M(y) dy < +\infty.$$

Then the set \mathcal{H}_M of all functions u harmonic in Ω and satisfying $|u(x, y)| \leq M(|y|)$, $(x, y) \in \Omega$, is uniformly bounded on any compact subset of Ω .

Key words and phrases. Harmonic functions, Levinson "log log" theorem.

This result has been proved by Dyn'kin in [6] by a different method under some stronger regularity conditions imposed on M . For any compact set $K \subset \Omega$ our approach provides an explicit estimate for $\sup_{u \in \mathcal{H}_M} \sup_K |u|$ in terms of M , K and Ω . We obtain Theorem 1.1 as a corollary of the "holomorphic" Levinson theorem by a reduction to axially-symmetric functions u . First, we prove the Theorem 1.1 in dimension 4, it implies the 3-dimensional case. Then we reduce the case of odd n to the case $n = 3$. The case of even n follows by adding the fake variable. The main obstacle, which appears in the higher-dimensional harmonic analog of the Levinson log log theorem, is the fact that $\log |\nabla u|$ is not necessarily subharmonic for a general harmonic function u in \mathbb{R}^n if $n \geq 3$.

Some of the proofs of the "holomorphic" Levinson log log theorem are of complex nature, some use implicitly or explicitly harmonic measure estimates in cusp-like domains, but most of the proofs require the monotonicity condition on M , except for the brilliant idea due to Domar (see [18],[7],[8]), which avoids any regularity assumptions on M , even the monotonicity. We will sketch Domar's proof in Section 2, and use it to obtain explicit uniform estimates for \mathcal{H}_M in higher dimensions. We don't know whether Theorem 1.1 is valid for arbitrary majorants M satisfying (1) (even for $n = 2$).

For any $x, y \in \mathbb{R}^n$ let $d(x, y)$ denote the Euclidean distance between x and y . For any $X, Y \subset \mathbb{R}^n$ we use the notation $d(X, Y)$ for $\inf\{d(x, y) : x \in X, y \in Y\}$. The symbol λ_n will denote the n -dimensional Lebesgue measure in \mathbb{R}^n .

2. DOMAR'S ARGUMENT

Theorem 2.1. *Let f be a holomorphic function in a rectangle $P := (-a, a) \times (-b, b)$. Suppose that a function $M(y)$ satisfies $\int_{-b}^b \log^+ \log^+ M(y) dy < +\infty$ and $|f(x + iy)| \leq M(y)$ for all $(x, y) \in P$. Then for any compact set $K \subset P$ there exists a constant $C = C(M, d(K, \partial P))$ such that $\sup_K |f| < C$.*

Theorem 2.1 immediately follows from the next lemma on subharmonic functions, since $\log |f|$ is subharmonic.

Lemma 2.2. *Let v be a subharmonic function in a rectangle $P := (-a, a) \times (-b, b)$. Suppose that a function \tilde{M} satisfies $\int_{-b}^b \log^+ \tilde{M}(y) dy < +\infty$ and $v(x + iy) \leq \tilde{M}(y)$ for all $(x, y) \in P$. Then for any compact set $K \subset P$ there exists a constant $C = C(\tilde{M}, d(K, \partial \Omega))$ such that $\sup_K v \leq C$.*

Sketch of the proof. Let $F(t) := \lambda_1(\{y \in (-b, b) : \tilde{M}(y) \geq t\})$ denote the complementary cumulative distribution function of $\tilde{M}(y)$. The logarithmic integral condition $\int_{-b}^b \log^+ \tilde{M}(y) dy < +\infty$ can be reformulated in terms of

F , namely $\sum_{i=0}^{+\infty} F(2^i) < +\infty$ if $\int_{-b}^b \log^+ \tilde{M}(y) dy < +\infty$ (see [18], p.378–379).

Then there exists a positive number C such that

$$(2) \quad \sum_{i=-1}^{+\infty} F(2^i C) < \frac{\pi}{8} d(K, \partial P).$$

Our aim is to show that $\sup_K v \leq C$. Assume the contrary. Suppose there is $z_0 \in K$ with $v(z_0) > C$. Let A_t denote the set $\{z \in P : u(z) \geq t\}$.

Proposition. *If a point $z \in P$ satisfies $v(z) \geq \mathcal{C}$ with $\mathcal{C} > 0$, and $d(z, \partial P) > \frac{8}{\pi} F(\mathcal{C}/2)$, then there is a $\zeta \in P$ such that $d(z, \zeta) \leq \frac{8}{\pi} F(\mathcal{C}/2)$ and $v(\zeta) \geq 2\mathcal{C}$.*

Consider the ball B centered at z with radius $r = \frac{8}{\pi} F(\mathcal{C}/2)$, note that $B \in P$, since $d(z, \partial P) > \frac{8}{\pi} F(\mathcal{C}/2)$. Now, the subharmonicity of v will be exploited: $\mathcal{C} \leq v(z) \leq \frac{1}{\lambda_2(B)} \int_B v = \frac{1}{\lambda_2(B)} \left(\int_{B \setminus A_{\mathcal{C}/2}} v + \int_{B \cap A_{\mathcal{C}/2}} v \right) \leq \mathcal{C}/2 + \frac{1}{\lambda_2(B)} \int_{B \cap A_{\mathcal{C}/2}} v$. Hence $\mathcal{C}/2 \leq \frac{1}{\lambda_2(B)} \int_{B \cap A_{\mathcal{C}/2}} v \leq \frac{1}{\pi r^2} \sup_B v \cdot \lambda_2(B \cap A_{\mathcal{C}/2}) \leq \frac{1}{\pi r^2} \sup_B v \cdot \lambda_1(\{x \mid \exists y : (x, y) \in B \cap A_{\mathcal{C}/2}\}) \cdot \lambda_1(\{y \mid \exists x : (x, y) \in B \cap A_{\mathcal{C}/2}\}) \leq \frac{1}{\pi r^2} \sup_B v \cdot 2r F(\mathcal{C}/2) = \frac{1}{4} \sup_B v$. Thus $2C \leq \sup_B v$ and the proposition is proved.

Using the proposition and taking z_0 in place of z and C in place of \mathcal{C} we obtain a point z_1 such that $v(z_1) \geq 2C$ and $d(z_1, z_0) \leq \frac{8}{\pi} F(C/2)$. Recall that $d(z_0, \partial P) > \frac{8}{\pi} \sum_{i=-1}^{+\infty} F(2^i C)$, hence $d(z_1, \partial P) > \frac{8}{\pi} \sum_{i=0}^{+\infty} F(2^i C)$. Exploiting the proposition infinitely many times we obtain a sequence $\{z_i\}_{i=0}^{+\infty}$ such that $v(z_i) \geq 2^i C$ and $d(z_i, z_{i+1}) \leq \frac{8}{\pi} F(2^{i-1} C)$. By (2) z_i have a limit point $z \in P$, hence $v(z) \geq \lim_{i \rightarrow \infty} v(z_i) = +\infty$, and a contradiction is obtained. \square

Remark 2.3. Domar's argument also provides explicit estimates in Theorem 2.1 of $C(M, d(K, \partial P))$. Define $F(t)$ by $F(t) = \lambda_1(\{y : \log^+ M(y) \geq t\})$. If $C > 0$ and $d(K, \partial P) > \frac{8}{\pi} \sum_{i=-1}^{+\infty} F(2^i C)$, then $|f| \leq \exp(C)$ on K .

3. AXIALLY-SYMMETRIC HARMONIC FUNCTIONS.

Consider $\mathbb{R}^n = \{x = (x_1, \dots, x_n) : x_i \in \mathbb{R}\}$. By ρ we denote $\sqrt{\sum_{i=1}^{n-1} x_i^2}$ and $h := x_n$. A function u defined in \mathbb{R}^n is called axially-symmetric if $u = u(\rho, h)$, i.e. u is invariant under orthogonal transformations of the first $(n-1)$ coordinates. An axially-symmetric harmonic function u satisfies the elliptic Euler-Darboux equation:

$$(3) \quad \frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial h^2} + \frac{n-2}{\rho} \frac{\partial u}{\partial \rho} = 0.$$

We are going to use two ideas. The first one reduces axially-symmetric harmonic functions in \mathbb{R}^4 to ordinary harmonic functions in \mathbb{R}^2 . The second trick reduces axially-symmetric harmonic functions in \mathbb{R}^{2k+3} to harmonic functions in \mathbb{R}^3 . It will help in dimension $n \geq 5$. We refer the reader to [1], [12], [13], [14], [16], [17], [23], [24], [28], [29] and references therein, where these and related ideas appear in a different context, however we are not able to locate their origin.

3.1. From \mathbb{R}^4 to \mathbb{R}^2 . Suppose u is an axially-symmetric harmonic function in an axially-symmetric domain $\Omega \subset \mathbb{R}^4$. Consider the set $\tilde{\Omega}_+ \subset \mathbb{R}^2$ defined by $x \in \Omega \iff (\rho(x), h(x)) \in \tilde{\Omega}_+$. It is easy to see from (3) that the function

$$(4) \quad \tilde{u}(\rho, h) = \rho u(|\rho|, h)$$

is harmonic in $\text{Int } \tilde{\Omega}_+$. Define $\tilde{\Omega}_-$ by $x \in \Omega \iff (-\rho(x), h(x)) \in \tilde{\Omega}_-$. Let $\tilde{\Omega}$ be the union of $\tilde{\Omega}_+$ and $\tilde{\Omega}_-$. Then $\tilde{\Omega}$ is a domain in \mathbb{R}^2 , symmetric with respect to the line $\rho = 0$. By the Schwarz reflection principle we see that (4) defines an odd (with respect to ρ) harmonic function in $\tilde{\Omega}$.

3.2. From \mathbb{R}^{2k+3} to \mathbb{R}^3 . Let $u = u(\rho, h)$ be an axially-symmetric harmonic function in \mathbb{R}^{2k+3} . Put

$$(5) \quad v(\varphi, \rho, h) = \rho^k e^{ik\varphi} u(\rho, h),$$

where (φ, ρ, h) are cylindrical coordinates in \mathbb{R}^3 . Then v is a harmonic (complex-valued) function in \mathbb{R}^3 . Indeed,

$$\Delta v = \frac{\partial^2 v}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial v}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 v}{\partial \varphi^2} + \frac{\partial^2 v}{\partial h^2} = 0 + \rho^k e^{ik\varphi} \left(\frac{\partial^2 u}{\partial \rho^2} + \frac{\partial^2 u}{\partial h^2} + \frac{2k+1}{\rho} \frac{\partial u}{\partial \rho} \right) = 0.$$

The last argument shows that v is harmonic in $\mathbb{R}^3 \setminus \{\rho = 0\}$. Note that v is continuous up to the line $\{\rho = 0\}$, which is a removable singularity for bounded harmonic functions (see [2], p.200). Thus v is harmonic in \mathbb{R}^3 .

4. PROOF OF THEOREM 1.1.

4.1. Proof of the case $n = 4$. Fix $\varepsilon > 0$: $R, H > \varepsilon$. Take any $x_0 \in \mathbb{R}^{n-1}$ with $|x_0| < R - \varepsilon$. Consider any function u from \mathcal{H}_M . It is sufficient to show that there is $C = C(M, H, \varepsilon)$ such that $|u(x_0, h)| \leq C$ for any h : $|h| < H - \varepsilon$. Denote the set $\{(x, y) : x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < \varepsilon, |y| < H\}$ by P_ε and consider the function $\tilde{u} : P_\varepsilon \rightarrow \mathbb{R}$ defined by $\tilde{u}(x, y) = u(x - x_0, y)$. Note that $|\tilde{u}(x, y)| \leq M(|y|)$ on P_ε .

Let us make an axial symmetrization step. Denote by $O(3)$ the group of orthogonal transformations in \mathbb{R}^3 , let dS be the Haar measure on $O(3)$. For any $g \in O(3)$ we use the notation \tilde{u}_g for the function $\tilde{u}(gx, y)$. It is clear that \tilde{u}_g is harmonic in P_ε , $\tilde{u}_g(0, y) = \tilde{u}(0, y) = u(x_0, y)$ and $|\tilde{u}_g(x, y)| \leq M(|y|)$ on P_ε . Put $w(x, y) := \int_{O(3)} \tilde{u}_g(x, y) dS(g)$, $(x, y) \in P_\varepsilon$, it is evident that w also enjoys the properties from the preceding sentence and $w = w(\rho, h)$ is axially-symmetric. We have reduced 4-dimensional case to the following lemma.

Lemma 4.1. *Suppose $w = w(\rho, h)$ is an axially-symmetric harmonic function in the truncated cylinder P_ε and $|w(x, y)| \leq M(|y|)$, then there is a constant $C = C(M, H, \varepsilon)$ such that $|w(0, y)| < C$ for any $y \in (-H + \varepsilon, H - \varepsilon)$.*

Proof. Put $v(\rho, h) := \rho w(|\rho|, h)$, by Section 3.1 v is harmonic in $(-\varepsilon, \varepsilon) \times (-H, H)$. Denote $\rho + ih$ by ζ , and $\frac{\partial v}{\partial \rho} - i \frac{\partial v}{\partial h}$ by f , then f is a holomorphic function in $(-\varepsilon, \varepsilon) \times (-H, H)$. Denote the set $(-\varepsilon/2, \varepsilon/2) \times (-H + \varepsilon/2, H - \varepsilon/2)$ by $\tilde{P}_{\varepsilon/2}$.

Take any $\zeta = (\rho, h) \in \tilde{P}_{\varepsilon/2}$ with $h \leq \varepsilon$ and consider a disk $B_{h/2}(\zeta) := \{z : |z - \zeta| < h/2\}$. Since $|u(\rho, h)| \leq M(|h|)$ and M is decreasing $\sup\{|v|(x) : x \in B_{h/2}(\zeta)\} \leq M(h/2)$. Applying standard Cauchy's estimates of derivatives of harmonic functions we obtain $|\nabla v|(\zeta) \leq C_1 \frac{\sup\{|v|(x) : x \in B_{h/2}(\zeta)\}}{h/2} \leq C_2 \frac{M(h/2)}{h}$, by C_1, C_2, C_3 we will denote absolute constants, whose value is less than 100. We note that $|f| = |\nabla v|$, hence $|f|(\zeta) \leq C_2 \frac{M(h/2)}{h}$.

If $\zeta \in \tilde{P}_{\varepsilon/2}$ with $h \geq \varepsilon$, then $B_{\varepsilon/4}(\zeta) \subset (-\varepsilon, \varepsilon) \times (-H, H)$. Using in a similar way Cauchy's estimates we obtain $|f(\zeta)| \leq C_3 \frac{M(h/2)}{\varepsilon}$. We therefore have $|f(\zeta)| \leq \max(\frac{100}{\varepsilon}, \frac{100}{h})M(h/2)$ for any $\zeta \in \tilde{P}_{\varepsilon/2}$. Denote $\max(\frac{100}{\varepsilon}, \frac{100}{h})M(h/2)$ by $\tilde{M}(h)$. It follows from the inequality $\log^+ a + \log^+ b + \log 2 \geq \log^+(a+b)$ that $\int_{-H}^H \log^+ \log^+ M(y) dy < +\infty$ implies $\int_{-H+\varepsilon/2}^{H-\varepsilon/2} \log^+ \log^+ \tilde{M}(y) dy < +\infty$.

Now, we are in a position to apply Theorem 2.1 to the function f holomorphic in $\tilde{P}_{\varepsilon/2}$ with the majorant \tilde{M} , that gives us a positive constant $C = C(M, H, \varepsilon)$: $|f(0, h)| < C$ for $h \in (-H + \varepsilon, H - \varepsilon)$. Recall that $v(\rho, h) = \rho \tilde{u}(\rho, h)$, it yields $|u(0, h)| = |v_\rho(0, h)| \leq |f| \leq C(M, H, \varepsilon)$.

Remark 4.2. Let $\tilde{F}(t)$ denote $\lambda_1(\{h \in (-H + \varepsilon/2, H - \varepsilon/2) : \max(\frac{100}{\varepsilon}, \frac{100}{h})M(h/2) \geq \exp(t)\})$, then $C(M, H, \varepsilon)$ can be given explicitly in terms of \tilde{F} in view of

Remark 2.3. Namely, if $\varepsilon/2 > \frac{8}{\pi} \sum_{i=-1}^{+\infty} \tilde{F}(2^i C)$ for a positive constant C , then $u(x, y) \leq \exp(C)$ for all (x, y) with $|x| \leq R - \varepsilon, |h| \leq H - \varepsilon$.

Remark 4.3. The 4-dimensional case of Theorem 1.1 implies the 3-dimensional one (as well as the 2-dimensional) because we can always add a fake coordinate to \mathbb{R}^3 .

4.2. Proof of the case $n \geq 5$. We will consider only the case of odd $n = 2k + 3$. Now, we know that Theorem 1.1 holds for $n = 2, 3, 4$. We will prove the case of odd $n = 2k + 3$ reducing it to the case $n = 3$ with the help of idea discussed in Section 3.2. The case of even n follows immediately. Like in the proof of 4-dimensional case we can perform the axial-symmetrisation step and Theorem 1.1 is reduced to the following lemma.

Lemma 4.4. *Suppose $u = u(\rho, h)$ is an axially-symmetric harmonic function in a truncated cylinder $P_\varepsilon = \{(x \in \mathbb{R}^{n-1}, y \in \mathbb{R}, |x| < \varepsilon, |y| < H)\}$ such*

that $|u(x, y)| \leq M(|y|)$. Then there is a constant $\mathcal{C} = \mathcal{C}(n, M, H, \varepsilon)$ such that $|u(0, y)| < \mathcal{C}$ for $y \in (-H + \varepsilon, H - \varepsilon)$.

Following Section 3.2 we consider a function v defined by $v(\varphi, \rho, h) = \operatorname{Re}(\rho^k e^{ik\varphi} u(\rho, h))$ on the set $\{\varphi \in [0, 2\pi), \rho \in [0, \varepsilon), h \in (-H + \varepsilon, H + \varepsilon)\}$, where v is harmonic. With the help of a 3-dimensional case of Theorem 1.1 we can obtain $|v(\varphi, \rho, h)| < C(M, H, \varepsilon/2)$ for $\varphi \in [0, 2\pi), \rho \in [0, \varepsilon/2), h \in (-H + \varepsilon/2, H - \varepsilon/2)$. Then for any $h \in (-H + \varepsilon, H - \varepsilon)$ and the ball B centered at the point $(0, 0, h)$ with radius $\varepsilon/2$ we have $\sup_B |v| \leq C(M, H, \varepsilon/2)$. Applying standard estimates of the higher derivatives of harmonic functions we obtain $\frac{\partial^k}{\partial \rho^k} v \leq C(k) \frac{C(M, H, \varepsilon/2)}{(\varepsilon/2)^k}$ on the set $\{\varphi \in [0, 2\pi), \rho = 0, h \in (-H + \varepsilon/2, H - \varepsilon/2)\}$, where $C(k)$ is a constant depending only on dimension ($n = 2k + 3$). Take $\varphi = \rho = 0$ and see that $\frac{\partial^k v}{\partial \rho^k}(0, 0, h) = k! u(0, h)$. Thus $|u(0, h)| \leq C(k) \frac{C(M, H, \varepsilon/2)}{(\varepsilon/2)^k}$ for $h \in (-H + \varepsilon, H + \varepsilon)$.

5. APPLICATION TO THE UNIVERSAL POLYNOMIAL EXPANSIONS OF HARMONIC FUNCTIONS.

Consider the unit ball $\mathbb{B} := B_1(0)$ in \mathbb{R}^n . Any function h harmonic in \mathbb{B} admits power series expansion $h = \sum_{n=0}^{+\infty} h_n$, where h_n is a homogeneous harmonic polynomial of degree n . It is said that h belongs to the collection U_H , of harmonic functions in B with universal homogeneous polynomial expansions, if for any compact set $K \subset \mathbb{R}^n \setminus \mathbb{B}$ with connected complement and any harmonic function u in a neighbourhood of K , there is a subsequence $\{N_k\}$ of \mathbb{N} such that $\sum_0^{N_k} h_n \rightarrow u$ uniformly on K . This class of universal functions has been studied in [22], [9], [10], [3]. The following statement improves Theorem 7 from [9] on the boundary behavior of functions from U_H .

Theorem 5.1. *Let $\psi : [0, 1) \rightarrow \mathbb{R}^+$ be an increasing function such that $\int_0^1 \log^+ \log^+ \psi(t) dt < +\infty$. If $h = \sum_{n=0}^{+\infty} h_n$ enjoys $|h(x)| \leq \psi(|x|)$ on $B_r(\omega) \cap \mathbb{B}$ for some $\omega \in \partial\mathbb{B}$ and $r > 0$, then $f \notin U_H$.*

We won't prove Theorem 5.1 here, because all necessary ingredients of the proof with one exception are given in [9], where Theorem 5.1 is proved under the stronger assumption $\int_0^1 \log^+ \psi(t) dt < +\infty$ in place of $\int_0^1 \log^+ \log^+ \psi(t) dt < +\infty$. The only missing ingredient in [9], which allows to replace one log by log log, is the "harmonic" analog of the Levinson log log theorem in higher dimensions (its version in a ball, which follows from Theorem 1.1 with the help of Kelvin transform).

6. A QUESTION ON ONE-SIDED ESTIMATES.

Suppose that z_0 is a point in a square $Q = (-1, 1) \times (-1, 1)$ and M is a positive (decreasing and regular) function on $(0, 1)$. Under what assumptions

on M the family F_M^+ of all functions f holomorphic in Q , and satisfying $\operatorname{Im}(f(z)) \leq M(|\operatorname{Im}(z)|)$, $f(z_0) = 0$ is normal in Q ?

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